MODULE

TEACHING OF MATHEMATICS

(Classes XI-XII)

For

Master Trainers / Teachers

(In-Service Training Programme)

DIRECTORATE OF CURRICULUM & TEACHER EDUCATION NWFP ABBOTTABAD

JANUARY, 2003

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DIRECTORATE OF CURRICULUM AND
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ABBOTTABAD

JAN – FEB, 2003

Title	**Module, Teaching of Mathematics Class XI-XII for Master Trainers /Teachers of In-Service Training Programme.**
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FOREWORD

Directorate of Curriculum & Teacher Education, NWFP, Abbottabad is launching a comprehensive programme of in-service through out the province for all subjects/categories for the classes 6th to 12th under the title "Teacher Training Programme" scheme Improvement of Learning Environment For Quality Improvement for the year 2002-2004 as per policy of the Govt of NWFP, School & Literacy Department, Peshawar. The prime focus of this manual is training delivery effectively. There are two approaches to teacher's professional development, the carporate approach and the individual one, but in this guide book attempts are made to link the both practically.

To make the INSET Programme more effective and successful a "Survey Study" has been conducted to collect the feed back, needs of the learners, requirements of the teaching staff and desires of the concerned managers through, interview/questionnaires, survey form and classroom observation forms. Sample for the study was selected a few middle and secondary/Higher Secondary schools (Girls boys urban & rural).

The study was conducted by the Deputy Director (Training) and Subject Specialists of this Directorate.

In the light of above information & facts training strategy and instructional material has been developed to improve the learning environment for quality improvement through the innovative methodology and pedagogical techniques.

Instructional material consists on training manual for lead trainers & field trainers for delivery of training effectively and modules for each subject (VI – XII/Science/Arts) to facilitate the field Trainers as well as trainees of all categories (SS, SET (Science/Arts), CT, AT, TT).

The training manual comprises two parts, one for Subject Specialists training imparted by PITE and the other one for SET/CT/AT/TT training imparted by RITEs NWFP.

Umar Farooq
Director
Curriculum & Teacher Education
NWFP, Abbottabad

Introduction:

To make the teaching of mathematics effective, systematic and interesting, new pattern (Active – learning) is adopted.
Where ever possible active learning is focused, which help the understanding of subject quickly.

Contents:

(i) Permutation
(ii) Area of a Triangle
(iii) Determinant and their properties.
(iv) Definite integral
(v) Arithmetic sequence

Objectives:

(i) Students must be trained to develop a suitable level of mathematical competence.
(ii) Students should be able to use basic mathematics technique (active – learning).
(iii) Through active learning and training in mathematical skills, the student will be able to develop a logical and systematic approach to the formulation and solution of problems of everyday life.

Mathematical Concepts and application have been carefully explained and motivated where ever possible.
Using activity based Method, examples are presented in detail.

Topics are selected which covers the hard areas of mathematical concepts. Every effect have been made to make the modules readable and systematic. It is strongly hoped that this module will be found stimulating and according to needs of students.

Suggestions for its improvements in quality content will be welcome.

S. Inayatur Rehman
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TOPIC: Determinants and their Properties

Objective:

After studying this lesson students will be able to:

1. Define and explain determinants
2. Find and calculate determinants
3. Explain the properties of determinants
4. Use / apply determinants and their properties

Material Required:

Suitable number of copies of work sheets, chalks, black board, duster.

Concept and Content:

If A is a square matrix over real numbers, then a real number associated with each square matrix is called determinant. If A is square matrix of order nxn, the associated determinant is called a determinant of order n .The notation |A | is used to denote the determinant of A.

$$\text{If} \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\text{Then} \quad |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Now let us consider 3 x 3 matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\Rightarrow \quad |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

So, by expansion, we get

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \longrightarrow \text{(i)}$$

This is known as expansion of |A | with the help of first row
Similarly if we expand the determinant with the help of first column, we will get the same result as above (i).

Properties of Determinants:

To evaluate the determinant, properties of determinant have great importance. Some properties are as follow:

1. If each element in any row or each element in any column of a square matrix A is 0 then whole the value of determinant will be zero i.e.
$$|A| = 0$$
2. If any row or column of a square matrix are interchanged, the determinant of resulting matrix is the additive inverse of the determinant of the original matrix.
3. If two rows or two column in a square matrix are identical, then the value of whole the determinant is zero.
4. If each of element of one row or one column of a square matrix is multiplied by K, then the determinant of resulting matrix is K time the determinant of the original matrix.
5. If each element in a row or column of a square matrix is written as the sum of two terms, then the determinant of the resulting matrix can be written as the sum of the two determinants.
6. If each element of one row or column of a square matrix is multiplied by a real number K and the resulting product is added to the corresponding element in an other row or column respectively in the matrix, then the determinant of the resulting matrix is equal to the determinant of the original matrix.

Examples ⟶ **How to use properties of Determinant:**

* **Prove that**

$$\begin{vmatrix} \beta\gamma & \alpha & \alpha^2 \\ \gamma\alpha & \beta & \beta^2 \\ \alpha\beta & \gamma & \gamma^2 \end{vmatrix} = \begin{vmatrix} 1 & \alpha^2 & \alpha^3 \\ 1 & \beta^2 & \beta^3 \\ 1 & \gamma^2 & \gamma^3 \end{vmatrix}$$

Where α, β, γ are non-zero elements.

Proof:

Let us consider the L.H.S
Operating $R_1 \times \alpha$, $R_2 \times \beta$ and $R_3 \times \gamma$, we get:

$$\frac{1}{\alpha\beta\gamma}\begin{vmatrix} \alpha\beta\gamma & \alpha^2 & \alpha^3 \\ \alpha\beta\gamma & \beta^2 & \beta^3 \\ \alpha\beta\gamma & \gamma^2 & \gamma^3 \end{vmatrix}$$

(by taking '$\alpha\beta\gamma$' common from c_1)

$$= \frac{\alpha\beta\gamma}{\alpha\beta\gamma} \begin{vmatrix} 1 & \alpha^2 & \alpha^3 \\ 1 & \beta^2 & \beta^3 \\ 1 & \gamma^2 & \gamma^3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \alpha^2 & \alpha^3 \\ 1 & \beta^2 & \beta^3 \\ 1 & \gamma^2 & \gamma^3 \end{vmatrix} = \text{R.H. S}$$

Hence proved

Expand the determinant

$$\begin{vmatrix} 4 & 5 & 6 \\ -3 & 1 & 2 \\ 1 & -2 & 3 \end{vmatrix}$$

Solution:

We will expand this determinant by the help of first row.
Therefore,

$$\begin{vmatrix} 4 & 5 & 6 \\ -3 & 1 & 2 \\ 1 & -2 & 3 \end{vmatrix}$$

$$= 4 \begin{vmatrix} 1 & 2 \\ -2 & 3 \end{vmatrix} -5 \begin{vmatrix} -3 & 2 \\ 1 & 3 \end{vmatrix} +6 \begin{vmatrix} -3 & 1 \\ 1 & -2 \end{vmatrix}$$

$= 4(3+4)-(-9-2)+6(6-1)$
$= 4(7)-5(-11)+6(5)$
$= 28+55+30$
$= 113$ Ans.

Note:

We can also expand by first column the answer will be same.(It means we can expand by help of any row or column).

Verify that

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)\,(b-c)\,(c-a)$$

Proof or verification:

Let us take L.H.S

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

Now subtracting the first row from second and third row respectively we get,

$$\begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$$

$$\begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & (b-a)(b+a) \\ 0 & c-a & (c-a)(c+a) \end{vmatrix}$$

Now taking (b-a) and (c-a) common from R_2 and R_3 respectively we have

$$= (b-a)\,(c-a) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix}$$

$$= (b-a)\,(c-a) \begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix}$$ (By expanding through first column)

$= (b-a)\,(c-a)\,(c+a - b-a)$

$= (b-a)\,(c-a)(c-b)$

$= (a-b)\,(b-c)\,(c-a)$ = R.H.S

Proved

Prove that

$$\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$$

Proof-

Adding C_2 and C_3 in C_1 we get

$$= \begin{vmatrix} a-b+b+c+c-a & b-c & c-a \\ b-c+c-a+a-b & c-a & a-b \\ c-a+a-b+b-c & c-a & b-c \end{vmatrix}$$

$$= \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix} \text{ Proved}$$

(Because by the property of determinant here C_1 is zero so whole the value of determinant is zero)

Proved that

$$\begin{vmatrix} a+l & a & a \\ a & a+l & a \\ a & a & a+l \end{vmatrix} = l^2(3a+l)$$

Proof

Let us consider L.H.S

$$\begin{vmatrix} a+l & a & a \\ a & a+l & a \\ a & a & a+l \end{vmatrix} \quad \text{(Expanding through first Row)}$$

$$=(a+l)\begin{vmatrix} a+l & a \\ a & a+l \end{vmatrix} -a \begin{vmatrix} a & a \\ a & a+l \end{vmatrix} +a \begin{vmatrix} a & a+l \\ a & a \end{vmatrix}$$

$$= (a+l) \left\{(a+l) - a\right\} - a \left\{a(a+l) - a^2\right\}$$
$$+a \left\{a^2 - a(a+l)\right\}$$

$=(a+l)\ (a^2+l^2+2al-a^2)-a\ (a^2+al-a^2)+a\ (a^2-a^2-al)$
$=(a+l)\ (l^2+2al)-a^2l-a^2l$
$=al^2+2a^2l+l^3-+2al^2-2a^2l$
$=3al^2+l^3$
$=l^2(3a+l)$=R.H.S Proved.

METHOLOGY

(Initiative) Activity No. 1:

1. Divide the class in to pair groups.
2. Distribute copies of work sheets to each group.
3. Give instruction to students to read the question carefully and answer the question as directed.
4. Guide and monitor the students discuss each other and then one by one.
5. Tell the students that they discuss each other and then given answers of questions asked.
6. Now collect the work sheets from each group.

NOTE:

A sample of work sheets is given on last page of module. (Page 17)

7. Now in the light of Answers given by students and approach of students, very carefully and with funny hand relate these to the topic to be delivered i.e. "determinant".
8. Tell the simple definition of determinants (Associated) with each square Matrix A over real numbers is a real number called the detriment of A. (Write the definition on blackboard) It is denoted by $|A|$.
9. Tell the students that if we have

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{Which is } (2 \times 2) \text{ matrix}$$

Then $|A| = a_{11}\,a_{22} - a_{12}\,a_{21}$

10. Now bend the attention of students to find the determinants of (3×3) matrix.
11. For this write a (3×3) matrix on black board i.e.

$$\text{If} \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

12. Before going forward, tell the students to understand following pattern i.e.

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

13. Now

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Activity No. 2:

1. Tell the students that determinations have many applications other than in the liner equation. Tell them that determinants have following main properties.

   a. The interchange of two rows or two columns changes the sign of the value of the determinant. e.g.

$$\begin{vmatrix} 3 & 0 & 2 \\ 1 & 1 & 5 \\ 2 & 1 & 3 \end{vmatrix} = - \begin{vmatrix} 3 & 2 & 0 \\ 1 & 5 & 1 \\ 2 & 3 & 1 \end{vmatrix}$$

   b. If two rows or two column of a determinant are identical, then value of the determinant is zero. e.g.

$$\begin{vmatrix} 3 & 5 & 2 \\ -4 & 6 & 9 \\ -4 & 6 & 9 \end{vmatrix} = 0 \quad \text{(Its } R_2 \text{ \& } R_3 \text{ are identical)}$$

   c. If any row or any column of a determinant is zero, then the value of the determinant will be zero. e.g.

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 2 & -9 & 5 \end{vmatrix} = 0 \text{ (As } R_2 \text{ is zero)}$$

Activity No. 3 (Application of Properties)

1. Write the following question on black board "Using the properties of determinant" Verify that

$$\begin{vmatrix} \beta\gamma & \alpha & \alpha^2 \\ \gamma\alpha & \beta & \beta^2 \\ \alpha\beta & \gamma & \gamma^2 \end{vmatrix} = \begin{vmatrix} 1 & \alpha^2 & \alpha^3 \\ 1 & \beta^2 & \beta^3 \\ 1 & \gamma^2 & \gamma^3 \end{vmatrix}$$

Solution:

2. Tell the students that we will take L.H.S. i.e.

$$\begin{vmatrix} \beta\gamma & \alpha & \alpha^2 \\ \gamma\alpha & \beta & \beta^2 \\ \alpha\beta & \gamma & \gamma^2 \end{vmatrix}$$

3. **Now operating $R_1 \times \alpha$, $R2 \times \beta$ and $R_3 \times \gamma$, we have**

$$=1/\alpha\beta\gamma \begin{vmatrix} \alpha\beta\gamma & \alpha^2 & \alpha^3 \\ \alpha\beta\gamma & \beta^2 & \beta^3 \\ \alpha\beta\gamma & \gamma^2 & \gamma^3 \end{vmatrix}$$

4. **Taking $\alpha\beta\gamma$ as common from C_1, we get ,**

$$=\alpha\beta\gamma/\alpha\beta\gamma \begin{vmatrix} 1 & \alpha^2 & \alpha^3 \\ 1 & \beta^2 & \beta^3 \\ 1 & \gamma^2 & \gamma^3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \alpha^2 & \alpha^3 \\ 1 & \beta^2 & \beta^3 \\ 1 & \gamma^2 & \gamma^3 \end{vmatrix} = \text{R.H.S.}$$

Activity No 4:

1. Ask the students to write/ Note the following question on Notebooks (First write question your self on black board).
2. Verify that

$$\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

Verification:

3. Tell the students that we will take first L.H.S .i.e.

$$\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$$

4. Now operating $R_1 \times a$, $R_2 \times b$, $R_3 \times c$, we get,

$$=1/abc \begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix}$$

$$= \frac{abc}{abc} \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$ (Taking 'abc' common from c_3)

$$= \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$ (Interchanging the columns twicely)

= R.H.S Proved

Activity No: 5

1. **Write the following on black board.**
   Verify that

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

Solution:

Let $\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

2. **Now subtracting R_1 from R_2 & R_3 we get,**

$$\Delta = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$$

$$\Rightarrow \quad \Delta = (b-a)(c-a) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix}$$

$$= (b-a)(c-a) \begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix}$$

$= (b-a)(c-a)(c+a-b-a)$
$= (b-a)(c-a)(c-b)$
$= (a-b)(b-c)(c-a)$ = R.H.S **Proved**

Summary:

(Summaries in following way)

Basically determinant is a value of matrix. If A is a square matrix of order (nxn), the associated determinant is called a determinant of order n.

The notation $|A|$ is used to denote the determinant of A.
Determinants have following main properties.

a. **The interchange of two rows or two columns changes the sign of the value of the determinant. e.g.**

$$\begin{vmatrix} 3 & 0 & 2 \\ 1 & 1 & 5 \\ 2 & 1 & 3 \end{vmatrix} = - \begin{vmatrix} 3 & 2 & 0 \\ 1 & 5 & 1 \\ 2 & 3 & 1 \end{vmatrix}$$

b. **If two rows or two column of a determinant are identical, then value of the determinant is zero. e.g.**

$$\begin{vmatrix} 3 & 5 & 2 \\ -4 & 6 & 9 \\ -4 & 6 & 9 \end{vmatrix} = 0 \quad \text{(Its } R_2 \text{ \& } R_3 \text{ are identical)}$$

c. **If any row or any column of a determinant is zero, then the value of the determinant will be zero. e.g.**

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 2 & -9 & 5 \end{vmatrix} = 0 \text{ (As } R_2 \text{ is zero)}$$

Self Assessment:

Assign the following task for home work.

Q No 1. **Prove that**

$$\begin{vmatrix} x+1 & x+2 & x+3 \\ x+4 & x+5 & x+6 \\ x+7 & x+8 & x+9' \end{vmatrix} = 0$$

Q No 2. **Expand (By first column)**

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & -1 & 0 \\ 0 & 1 & 0 \end{vmatrix}$$

Q No 3. **Show that**

$$\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = 0$$

Q No 4. **Mark true or false.**

(i) **We can not find the determinant of rectangular matrix. (T/F)**

(ii) **If any row or any column in a determinant is zero, then whole value determinant will be zero. (T/F)**

(iii) **The interchangingof two rows or two columns changes the sing of the determination. (T/F)**

Answer **(i) T (ii) T (iii) T**

Sample of work Sheet:

Q No. 1 Define square matrix.

Q No. 2 What is meant by the order of a matrix.

Q No. 3 What do you mean by a real number.

Q No. 4 If

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 5 \end{bmatrix} \quad \text{(2x2 matrix)}$$

Then $|A| = ?$

Ans. 3

Q No. 5 If $A = \begin{vmatrix} 1 & 5 \end{vmatrix}$

Can we find $|A|$ (Yes / No)
(Mark Yes or No)

Q No. 6 To make situation interesting

Ask

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ (3x3)

Then find $|A| = ?$

NOTE: Q NO 6. is just to invite the students to your today topic i.e. Determinant of 3x3 matrix.

NOTE FOR TEACHER:

As material wise and physically, the topic is lengthy and heavy, so it is difficult to complete it in a single educational period.

Teacher may continue it on next day.

Topic # Arithmetic Sequence

Objectives:

After studying this lesson students will be able to.

1. Define and explain arithmetic sequence.
2. Find nth term of any arithmetic progression (A.P)
3. Use / apply arithmetic sequence.

Material required:

Small balls (suitable numbers), black board, chalk, Duster.

Concept and Content:

Arithmetic sequence is defined as " A sequence every term of which after the first is obtained from the preceeding term by adding a fixed number is called an arithmetic sequence or arithmetic progression denoted by (A.P).

Basically a sequence in an arrangements of number formed according to some definite rule. In mathematics we have to deal with number of sequences. Arithmetic sequence is one of them.

In the field of sets, a sequence is a function whose domain is a sub set of natural numbers.

For example, $S(n) = n^{2,}\ n \in N$ is a sequence function.

Now consider the sequence of number 2,4,6,8,............... each term after the first is obtianed by adding a fixed number 2 to the preceding term. This sequence is an example of arithmetic sequence(AP).

The constant difference between any two consecutive terms of an AP is called "Common Difference". It is denoted by'd'.

In above example, $d=2$.

General Term of AP:

The standard or general form of an AP is

$a, a+d, a+2d, a+3d, \ldots\ldots$

First term $= a$

Common difference $= d$

In this case, $a_1 = a = a+(1-1)d$

$a_2 = a+d = a+(2-1)d$

$a_3 = a+2d = a+(3-1)d$

⋮

Therefore,

$$\boxed{a_n = a+(n-1)d}$$

Which is the general term of an AP.

Examples ⟶ **How to use Arithmetic sequence:**

Find the number of terms of an AP in which

$a_n = 130$, $d = 25$, $a = 5$

<u>Solution</u>

We have given $a_n = 130$, $d = 25$ and $a = 5$

We know that the general term of an AP is, $a_n = a+(n-1)d$

Let us put values, we get;

$130 = 5+(n-1)25$

$\Rightarrow$ $130 = 5+25n-25$

$\Rightarrow$ $130 = 25n-20$

$\Rightarrow$ $25n = 150$

$\Rightarrow$ $n = \dfrac{150}{25}$

$\Rightarrow$ $n = 6$ Ans

Example:

Find the first four terms of an AP in which $a = 7$, $d = -4$

Solution:

Here, $a_1 = 7$, $d = -4$

We know that, $an = a+(n-1)d$ (nth term of AP)

So,

$$a_2 = a+(2-1)d$$

$$\Rightarrow a_2 = 7+1(-4)$$

$$\Rightarrow a_2 = 3 \quad \text{and } a_3 = a+(3-1)d$$

$$\Rightarrow a_3 = 7+2(-4)$$

$$\Rightarrow a_3 = -1$$

Now, $a_4 = a + (4-1)d$

$$\Rightarrow a_4 = 7+3(-4)$$

$$\Rightarrow a_4 = 7-12$$

$$\Rightarrow a_4 = -5$$

Thus first four terms will be

7, 3, -1, -5

....

Example:

Find the 20th term of AP whose 3rd term is 7 and 8th term is 17.

Solution:

Given data

$a_3 = 7$, $a_8 = 17$

$a_{20} = ?$

We know that the nth term of an AP is given as

$$an = a+(n-1)d$$

So, $a_3 = a+(3-1)d$

But $a_3 = 7$ So, $7 = a+2d$ ....(i)

Similarly $a_8 = a+(8-1)d$

$\Rightarrow \quad 17 = a + 7d$ ....(ii)

Now subtracting (i) and (ii) we get

$$\begin{array}{l} 7 \quad a + 2d \\ 17 = +a + 7d \\ \hline -10 = -5d \end{array}$$

$$\Rightarrow \quad d = 2$$

Therefore putting, $d = 2$ in (i), we get

$7 = a + 2(2)$

$\Rightarrow \quad 7 = a + 4$

$\Rightarrow \quad a = 3$

Now we need a_{20},

According to nth term,

$a_{20} = a + (20-1)d$

$\Rightarrow \quad a_{20} = 3 + 19\ (2)$

$\Rightarrow \quad a_{20} = 3 + 38$

$\Rightarrow \quad a_{20} = 41$

Thus the 20th term of AP will be 41.

Methodology

Activity No 1: (Initiative)

1. Divide the class into suitable groups according to the strength of students.
2. Make group leader of each.
3. Provide two small balls to 1st group, four to 2nd group, six balls to 3rd group and eight balls to 4th group and so on.
4. Ask the leaders of group to tell the number of balls provided.
5. Write these numbers on black board.
6. Arrange these number according to 1st , 2nd, 3rd and 4th group i.e.

   2,4,6,8,………………..

7. Tell the students that this arrangement is called arithmetic sequence and tell simple definition of arithmetic sequence "A sequence in which every term is obtained after the first term by adding a fixed number in preceding term is called arithmetic sequence. It is denoted by (A.P).
8. Tell the students and in above arrangement first term = 2, 2nd term =4, 3rd term =6 and these are denoted by a_1 or a, a_2, a_3 ,a_4, and so on receptively.
9. Ask the students to tell the difference of number of balls in any two consecutive groups.
10. Write this difference on black board and tell the students that mathematically this difference is denoted by d and it is calculated as (2nd term – first term) i.e. $a_2 - a_1 = d$ or from any other two consecutive terms.
11. Ask the students to tell that how many groups have been provided the balls.
12. Write this number on black board and tell the students that this number shows the number of terms in arithmetic sequence or progression.

    i.e. n = no. of terms

    Now tell the students that mathematically there is very important term which is known as nth term and its general formula is

    $$\boxed{a_n = a + (n-1)d}$$

    Where a = first term

    d = common difference

    n = no. of terms

Activity No 2.

1. Write the number of balls provided to 1st, 2nd and 3rd groups respectively on black board i.e.

   2,4,6

2. Ask the students to write these numbers on note book and ask them to find next term, using the general terms formula of (AP).
3. Guide and monitor the groups.
4. Ask each group to present their work and make necessary correction if needed and encourage the groups.

Activity No 3.

1. Ask the students to find the indicated term of A.P 7,11,15,............... (a_7)
   Write on black board.
2. Tell the students that here first term is equal to 7. Guide the students to find the difference i.e. 11-7 = 4
3. Call a role number of a student and ask him to tell general term of arithmetic sequence.
4. Write this formula on black board i.e.

   $$a_n = a + (n-1)d$$

5. Tell the students that in the question we have given $a = 7$ and $d = 4$ and we have to find a_7.

   So, $n = 7$

6. By applying nth term formula.

   We can say $a_7 = a + (7-1)d$

   $\Rightarrow \quad a_7 = 4 + 6(4)$

   $\Rightarrow \quad a_7 = 4 + 24 = 28$ Ans.

---

* Activity No 4.

1. Divide the class into suitable groups according to the strength of students.
2. Ask the students to find: Which term of the A.P 4,1, -2, ... is –77?
   Write above question on black board.
3. First of all given the hint to students, that here we will find n

4. Tell the students to write data i.e.

$a = 4,$

$d = 1\text{-}4$ (i.e. 2nd term – Ist term)

$= -3$

and an $= -77$

5. Guide them that here in this question,

$a_n = -77$

But $a_n = a+(n-1)\,d$

So $-77 = 4 + (n-1)(-3)$

$\Rightarrow\ -77 = 4 - 3n + 3$

$\Rightarrow\ -77 = 7 - 3n + 3$

$\Rightarrow\ -77 = 7 - 3n$

$\Rightarrow\ -77 - 7 = -3n$

$\Rightarrow\ -3n = -84$

$\Rightarrow\ n = -84/-3$

$= 28$ Ans.

* SUMMARY / MAIN POINTS:

Arithmetic sequence has the following important points to be noted.

1. It is always obtained by adding a fixed number (called common difference) in preceding term.
2. The general term of an arithmetic sequence is $a_n = a + (n-1)d$

   Where n = no. of terms

   d = common difference

   a = first term
3. If any three terms in the above formula are known, we can find the fourth one.
4. For verification of any sequence to be arithmetic sequence, we have to check the common difference i.e. if the common difference of 1st & 2nd, 2nd & 3rd and 3rd & 4th term are same, then the sequence will be arithmetic.

* **Self Assistant: (Assign The Following Task For Home Work)**

Q.No1. **Define the Arithmetic sequence.**

Q.No2. **What is general term of an (A.P).**

Q.No3. **Find the next two terms in A.P**

$$2x + 1, 2x + 4, 2x + 7$$

Q.No4. **Find the sequence whose general term is $3n - 1$.**

Q.No5. **Mark True and False.**

(i) The common difference is calculated by (First term – 2nd term). (T/F)

(ii) The common difference is always positive. (T/F)

(iii) A sequence in which every term after the first term is obtained by subtracting a fixed number from preceding term is called Arithmetic sequence . (T/F)

(iv) The general term of A.P is

$$a_n = a + (n-1)d \qquad \text{(T/F)}$$

(v) The first term in an (A.P) is always 1. (T/F)

Ans. (i) F (ii) F (iii) F (iv) T (v) F

Topic: Permutation

Objectives:

After studying this lesson students will be able to:

1. Define and explain permutation.
2. Find permutations.
3. Use / apply permutation.

Material Required:

Pieces of chalks, Models of triangle, rectangle and circle, made of hard paper, black board, and duster.

Concept and Content:

An arrangement of a finite number of objects some or all at a time is called a permutation of these objects. We can also define it as " An arrangement of set of n object in a given order is called a permutation of the object (taking all at a time).

OR

An arrangement of any $r \leq n$ of these objects in a given order is called a permutation of n objects taken r at a time. The number of permutation of n objects taken r at a time as denoted by P(n,r) or ^nPr.

Example:

Let S = {1,2,3,4}

Here order Pairs (1,2), (1,3), (1,4), (2,4) and triplet, (1,2,3),(1,3,2),(1,2,4) etc are permutation of S taken 2 at a time and 3 at a time.

....

Example:

How many numbers consisting of two digits can be formed 2,3,5,7. Each integer is to be used only once?

Solution:

The following numbers of two digits can be formed.

23	32	52	72
25	35	53	73
27	37	57	75

$\Rightarrow$ It means there are twelve possible formations consisting of two digits.

Example:

How many arrangements can be made of four letters a,b,c,d taken three at time.

Solution:

The following arrangements can be made according to given condition:

abc	acb	abd	adb	acd	adc
bcd	bdc	bac	bca	bad	bda
cda	cad	cab	cba	cbd	cdb
dab	dba	dbc	dcb	dac	dca

Evaluation of ^nPr:

We know that ^nPr stands for number of permutations of n different objects taken "r" at a time.

Let us suppose that we have n objects and we want to determine the number of ways in which these objects can be arranged by taking "r" at a time. Any one of the n objects can fill up the first place, so there are n different ways of filling the first place.

Now when the first placed has been filled up in any one of the n ways, (n-1) objects remain for the filling he second place. Therefore to each way of filling the first place there are (n-1) ways of filling the second place.

Hence the first two places can be filled up in n(n-1)ways .When the first two places are filled up in any one of the n(n-1) ways, there will be (n-2) objects left over .So for each

way of filling the first two places ,there will be (n-2) ways of filling the 3rd place. Hence first three places can be filled up in n(n-1)(n-2) ways.

Consider the following table.

Ways	n	(n-1)	(n-2)	------	n-r + 1
Place	1,st	2nd	3rd	-------	rth

So proceeding this way or in this manner, we can finally fill the r.th place in(n-r+1) ways
So, we can write

$^nP_r = n(n-1)(n-2) \ldots\ldots\ldots\ldots\ldots\ldots (n-r+1)$
Now when r = n, then
$^nP_n = n(n-1)(n-2) \ldots\ldots\ldots\ldots\ldots\ldots 3,2,1$
let say $n(n-1)(n-2) \ldots\ldots\ldots 3,2,1 = n!$
then $^nP_n = n!$

$$\text{also } ^nP_r = \frac{n(n-1)(n-2) \ldots\ldots(n-r+1)(n-r)(n-r-1) \ldots\ldots 3,2,1}{(n-r)(n-r-1) \ldots\ldots 3,2,1}$$

$$\Rightarrow \boxed{^np_r = \frac{n!}{(n-r)!}}$$

Application of above formula:

Example

How many distinct four digits number can be formed from the integers 1,2,3,4,5,6 if each integer is used only once?

Solution:

Here the total number of digits is 6,i.e. n = 6 and each number consist of four digits So, r = 4

So using. $${}^{n}p_{r} = \frac{n!}{(n-r)!}$$

$$\Rightarrow \quad {}^{6}p_{4} = \frac{6!}{(6-4)!}$$

$$\Rightarrow \quad {}^{6}p_{4} = \frac{6!}{2!}$$

$$\Rightarrow \quad {}^{6}p_{4} = \frac{6!}{2!}$$

$$\Rightarrow \quad {}^{6}p_{4} = \frac{6X5X4X3X2X1}{2X1}$$

$$\Rightarrow \quad {}^{6}p_{4} = 369 \qquad ANSWER$$

Example:

Find the number of arrangements of eight "8" distinct book on a shelf taken

(i) all at a time

(ii) 2 at a time?

Solution:

(i) Here $r = 8$ and

also $n = 8$

So using $^{n}P_{r} = \dfrac{n!}{(n-r)!}$

$\Rightarrow \quad ^{8}P_{8} = \dfrac{8!}{(8-8)!}$

$\Rightarrow \quad ^{8}P_{8} = \dfrac{8!}{0!}$

$\Rightarrow \quad ^{8}P_{8} = \dfrac{8!}{1}$

$\Rightarrow \quad ^{8}P_{8} = 8x7x6x5x4x3x2x1$

$\Rightarrow \quad = 40320$ Ans.

(ii) Here $n = 8$, and $r = 3$

So, $^{8}P_{3} = \dfrac{8!}{(8-3)!}$

$\Rightarrow \quad ^{8}P_{3} = \dfrac{8x7x6x5!}{5!}$

$\Rightarrow \quad ^{8}P_{3} \quad 336$ Answer

Example

How many signals can be given with four different flags when any number of them may be hoisted at a time?

Solution:

The number of flags = 4

So, number of signals using one flag = ${}^4P_1$ =4

Number of signals using 2 flags = ${}^4P_2$ = 12

Number of signals using 3 flags = ${}^4P_3$ = 24

Number of signals using 4 flags = ${}^4P_4$ = 24

Therefore, the total number of signals is = 4+12+24+24 = 64 Ans.

METHOLOGY

Activity No: 1 (Initiative):

1. Divide the class in to suitable groups according to the strength of students,
2. Make group leader of each.
3. Distribute one triangle, one rectangle made of hard paper to each group.
4. Tell the students to make columns A,B with chalks on their desks in following way

Column A	Column B

5. Now tell the students to put the triangles in column A and rectangle in column B.
6. Check and monitor each group.
7. Now ask the students to change the order of models i.e. guide them to put the triangle in column B and rectangle in column A, also demonstrate this on black board in following way

	Column A	Column B
first way ⟶	Triangle	Rectangle
2^{nd} way ⟶	Rectangle	Triangle

8. Now it is the suitable time to tell the students that there are only two ways to make them in order the models (i.e. Triangles and Rectangles) and tell them that this arrangement is called permutation and then tell the simple definition of permutation i.e

   "An arrangement of a finite number of objects some or all taken at a time is called permutation."
9. Now it is exactly the time to make the concept of permutation more strong. So, for this purpose distribute one more model i.e. circle made of hard paper and ask the students to make three columns on their desks.

10. **Demonstrate this your self on black board and guide the students.**

Column A	**Column B**	**Column C**

11. **Ask the students to put the models in column A,B,&C, through demonstration help the students in following way.**

First way $\longrightarrow$ TRC
2^{nd} way $\longrightarrow$ TCR
3^{rd} way $\longrightarrow$ RTC
4^{th} way $\longrightarrow$ RCT
5^{th} way $\longrightarrow$ CRT
6^{th} way $\longrightarrow$ CTR

where T = Triangle
R = Rectangle
C = Circle

Black Board Activity

12. **Now tell the students that when there are three models, we see that there are six ways to order the models (when all the object i.e. model are taken) and tell the students that in above activity three models were taken so, n= 3 and also all the models were taken at a time so, r = 3 also**
13. **Keeping this in mind, tell the students that mathematically permutation is denoted by P(n,r) or ^nP_r .**

<u>Activity No 2:</u>

Evaluation of ^nP_r (write on black board):

1. **Tell the students to open their notebooks and tell them to note important points in the evaluation of the formula of permutation.**
2. **Tell the students that ^nPr stands for number of permutation of n different objects taken 'r' at a time.**

3. Make table on black board i.e.

Ways →	n	n-1	n-2	………..	(n-r+1)
Places →	1,St	2nd	3rd	………..	r,th

4. Tell the students to make this table on their note books and clear the format of table, tell the students that we can fill the first place in n ways similarly we can fill the 2nd place in (n-1) ways and so on proceeding in this manner, we can finally fill the r,th place in (n-r+1) ways.

Tell the students that by fundamental principal, we can write

$^nP_r = n(n-1)\,(n-2)\ \ldots\ldots\ 3,2,1$

let write $n! = n(n-1)\,(n-2)\ \ldots\ldots\ 3,2,1$

thus tell the students (to note on their notebook)

$^nP_n = n!$

also $^nP_r = \dfrac{n(n-1)(n-2)\ \ldots\ldots(n-r+1)(n-r)(n-r-1)\ \ldots\ldots\ 3,2,1}{(n-r)(n-r-1)\ \ldots\ldots 3,2,1}$

$$\Rightarrow \boxed{\,^np_r = \frac{n!}{(n-r)!} \quad \boxed{\text{Required Formula}}\,}$$

Activity No 3:

1. Divide the class in to suitable group.
2. Ask the students to write the following question on their note books.
3. Write on black board the following question i.e.
   "How many distinct four digit numbers can be made from the integers 1,2,3,4,5,6, if each integer is used only once?
4. Tell the students here the total number of digits is 6 and i.e. $n = 6$ and as the each integer is used only once so tall them that here $r = 4$
5. Ask the students to tell the formula of permutation.
6. Write this formula on black board i.e. $^nP_r = \frac{n!}{(n-r)!}$

   By putting values in this formula

   $^6P_4 = \frac{6!}{(6-4)!}$

   $\Rightarrow \quad ^6P_4 = \frac{6x5x4x3x2x1}{2x1}$

   $\Rightarrow \quad ^6P_4 = 360$ Ans.
7. Tell the students to write the solution on notebooks.

Activity No 4:

1. Write the following question on black board.
   Find the number of arrangements of 8 digits books on a self-taken
   i. All at a time.
   ii. 3 at a time.
2. First of all write the formula of permutation i.e. $^nP_r = \frac{n!}{(n-r)!}$
3. Tell the students that here there are 8 distinct book, So number of arrangements of 8 digits book will be represented by n and in first part as all the books are taken at a time So, $r = 8$ also
4. Use the formula $^nP_r = \frac{n!}{(n-r)!}$

   Putting the values i.e.

   $^8P_8 = \frac{8!}{(8-8)!}$

   $\Rightarrow \quad ^8P_8 = \frac{8!}{0!}$

   $\Rightarrow \quad ^8P_8 = 8x7x6x5x4x2x1$

   $\Rightarrow \quad ^8P_8 = 40320$
5. Here the books on a self taken 3 at a time so tell the students that here $r = 3$ so by formula

   $\Rightarrow \quad ^8P_8 = \frac{8!}{(8-3)}$

   $\Rightarrow \quad ^8P_8 = \frac{8x7x6x5!}{5!}$

   $\Rightarrow \quad ^8P_8 = 336$

6. Ask the students that how much they are taking if a student or students face any problems, then guide and monitor the students carefully and make correction if needed.

Activity No 5:

1. On black board write the question.
   " How many different words can be found the letters of the words "BOXER" when
   i. All the letters are taken at a time.
   ii. Three letters are taken at a time.
   iii. O and X occupy the first and last places respectively.
   iv. B is kept in the middle.
2. Ask the students to tell how many letters are there in Word BOXER.
3. Write the number of letters on black board i.e. $r = 5$
4. Tell the students that as all the letters are taken at a time, So $r = 5$
5. Use the permutation formula

   $${}^{n}Pr = \frac{n!}{(n-r)!}$$ To find

   The different words that can be formed. i.e.

   $${}^{5}p_5 = \frac{5!}{(5-5)!}$$

   $$\Rightarrow \quad {}^{5}p_5 = \frac{5!}{(5-5)!}$$

6. As three letters are taken are taken at a time so tell them that here $r= 3$
   Therefore using permutation formula required words that cab be formed will be

   $${}^{5}p_5 = \frac{5!}{(5-3)!}$$

   $$\Rightarrow \quad {}^{5}p_5 = \frac{5!}{2!}$$

   $$\Rightarrow \quad {}^{5}p_5 = \frac{5x4x3x2x1}{2\times1}$$

   $${}^{5}p_5 = 60 \text{ Ans.}$$

7. Tell the students that when O and X occupy the first and lost places respectively, then the remaining three letters can be permuted among themselves i.e.

   $3! = 6$ ways

   Thus total number of words = 6

   8. Now when B id fixed in the middle the remaining 4 letters can be permuted among themselves in $4! = 24$ ways

      Thus to the number of words = 24

Summary: Summaries in following way.

An arrangement of a finite number of objects some or all taken at a time is called permutation. We can also define a permutation of "n" objects taken "r" at a time as follows:

'An arrangement of any $r \leq n$ of these objects in a given order is called a permutation of "n" it is denote by ^nPr.

To illustrate this let us consider the set S = {1,2,3,4}

From the set the order pair

(1,2,3)(1,3,2)(1,2,4) etc are permutation elements of S taken 2 at a time and three at a time.

Note for Teacher:

As material wise and physically the topic lengthy and heavy so it is difficult to complete it in one educational period.

Teacher may continue it on next day.

Self Assessment:

Assign the following task for H/W

Q No: 1 Define permutation.

Q No: 2 What is mathematical formula to calculate the permutation.

Q No: 3 How many words can be formed from the letters of the word "ARTICLE" using all the letters?

Q No: 4 From three books of Urdu four of Islamiyat and six of Pakistan Study how many sets of books can be chosen so that each set has two books in different subjects?

Q No: 5 make True and False.

(i) Six arrangements can be made from three pictures taken all at a time (T/F).

(ii) $^nPr = \frac{n!}{(r-p)}$ (T/F)

(iii) If r =n then $^nPn = n!$ (T/F)

(iv) If r =0 then $^nPo = 1$ (T/F)

(v) 5! = 121 (T/F)

Answers of Q No: 5

i. T ii. F iii. T iv. T v. F

Topic: Area of a Triangle

Objectives:

After studying this lesson students will be able to:

1. Define area of a Triangle.
2. Calculate area of any Triangle by different formula.
3. Use/apply the formula of area of any Triangle.

Material required:

Suitable number of Triangles, made of the hard papers, chalks, and Black board.

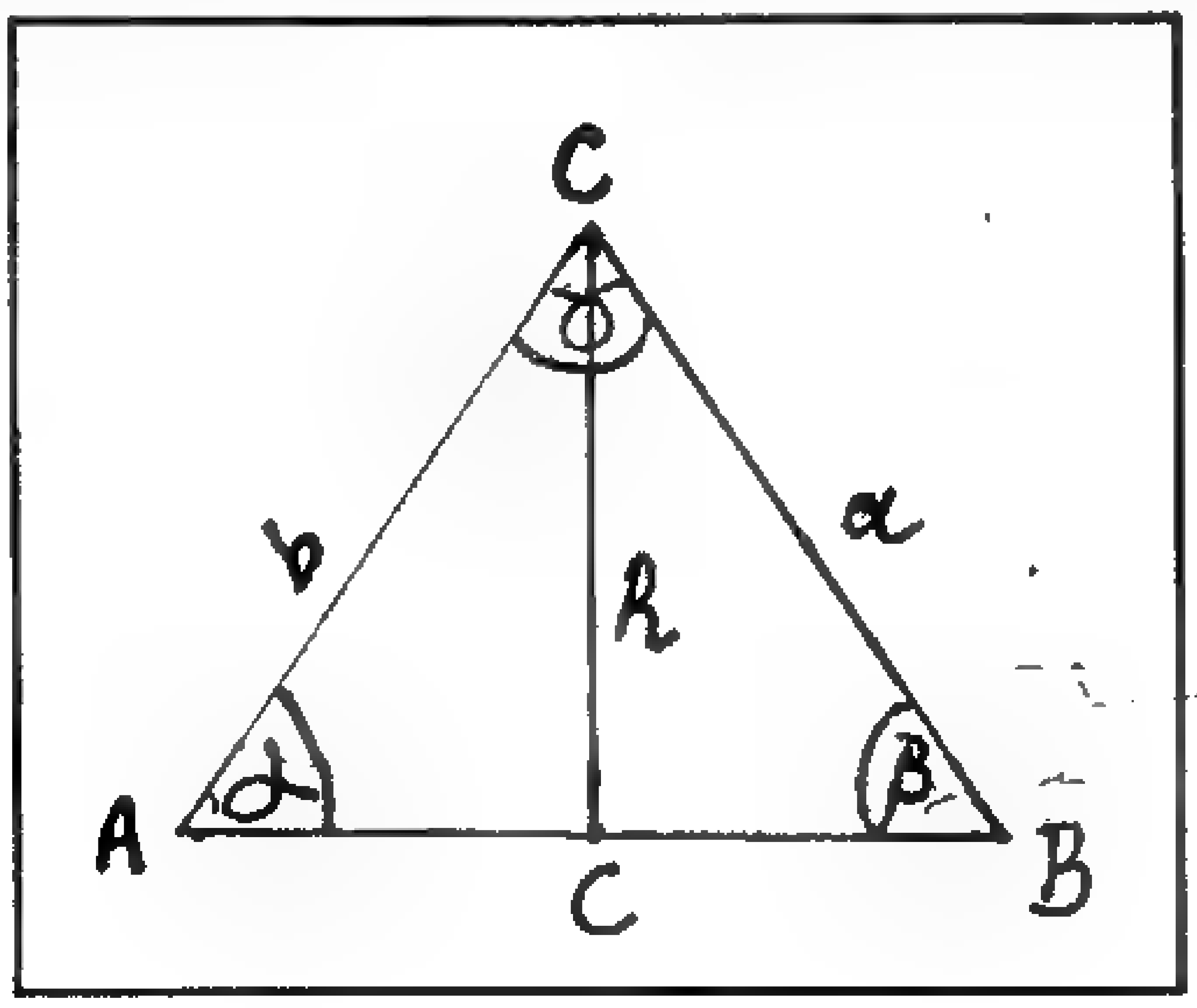
Concept and Content:

From elementary geometry we know that the area of a Triangle is equal to one half of product of measures of base and altitude.
Mathematically,

Area of $\triangle$ = ½ (base x altitude)

But it comes very interesting when area is required for any type of Triangle. i.e. Whether it is right angled triangle, acute angled triangle or obtuse angled triangle.

For this purpose let us consider the following figure.



If area is denoted by Δ then from above triangle ABC, we can say that

Area of ABC = ½ (measure of base x measure of altitude)

$\Rightarrow \quad \Delta = \frac{1}{2} c.h \longrightarrow$ (i)

But we know that $\sin\alpha = h/b$ and

$\sin \beta = h/a$

So, $h = b \sin\alpha$ and $h = a \sin \beta$

So (i)$\rightarrow$ $\Delta = \frac{1}{2}$ bc sinα

Or $\Delta = \frac{1}{2}$ ac sinβ

Or $\Delta = \frac{1}{2}$ ab sinγ

Therefore it is obvious from above formula that area of a Triangle is one half the product of measures of two sides and sine of angle between these two sides.

* **Similarly when measure of one side and the measure of two angles are given then area will be given as:**

$$\Delta = \frac{1}{2}a^2 \frac{\sin\beta \quad \sin\gamma}{\sin\alpha}$$

$$\Delta = \frac{1}{2}c^2 \frac{\sin\alpha \quad \sin\beta}{\sin\gamma}$$

and $$\Delta = \frac{1}{2}b^2 \frac{\sin\gamma \quad \sin\alpha}{\sin\beta}$$

* **Area of a Triangle when measure of all the sides are given:**

$$\Delta = \sqrt{S\ (S-a)\ (S-b)\ (S-c)}$$

Where S = ½ (a+b+c)

This formula is also known as " Hero's" formula.

<u>Examples</u> $\longrightarrow$ How to find the area of a Triangle.

* **Find the area of triangle ABC in which a = 756.26, c = 543.35, β = 47° 17′**

<u>Solution:</u>

Here we have given two sides and one-angle between theses two sides, So we will use

Δ = ½ ac sinβ

$\Rightarrow$ $\Delta = \frac{1}{2}(756.52)\ (543.35)\ \sin 47° 17'$

$\rightarrow$ Δ = 151004.66 square units

* Find he area of a Triangle in which

$a = 52.342,\ b = 29.572,\ c = 43.751$

Solution:

Here all the measure of three sides are given, so we will use Hero's formula i.e.

$$\Delta = \sqrt{S\ (S-a)\ (S-b)\ (S-c)}$$

First we calculate $S = \frac{1}{2}\ (a+b+c)$

$$\Rightarrow \quad S = \frac{1}{2}\ (52.342 + 29.572 + 43.751)$$

$$\Rightarrow \quad S = 62.8325$$

Now $S - a = 10.4905$

And $S - b = 33.2595$

And $S - c = 19.0815$

Therefore,

$$\Delta = \sqrt{(62.8325)\ (10.4905)\ (33.2595)\ (19.0815)}$$

$$\Rightarrow \quad \Delta = 646.78 \text{ square units}$$

Example: Find the area of the triangle in which $a = 22.24,\ c = 32°\ 15',\ \gamma = 65°\ 37'$

Solution:

We know that

$$\Delta = \frac{1}{2} a^2 \frac{\text{Sin}\,\beta \quad \text{Sin}\,\gamma}{\text{Sin}\,\alpha}$$

Now $\alpha + \beta + \gamma = 180°$

$\Rightarrow \quad \alpha = 180° - (\beta + \gamma)$

$\Rightarrow \quad \alpha = 180° - (32°15' + 65°37')$

$\Rightarrow \quad \alpha = 82°8'$

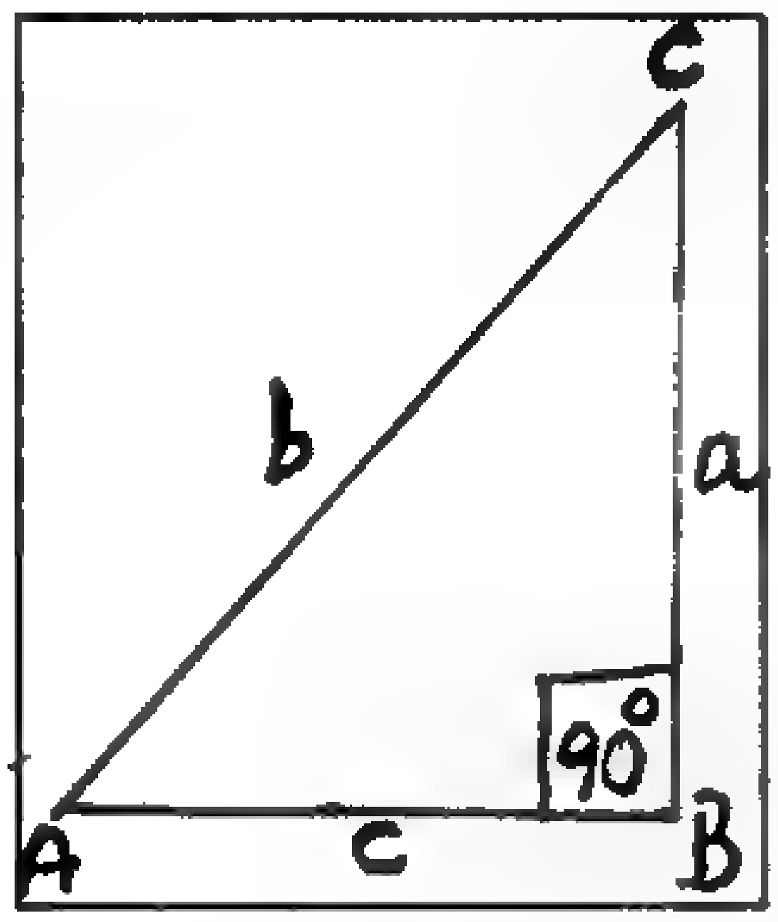
Therefore, $$\Delta = \frac{1}{2}(22.24)^2 \frac{\sin 32°15' \sin 65°37'}{\sin 82°8'}$$

$$\rightarrow \quad \Delta = 121.34 \text{ square units}$$

METHODOLOGY

Activity No: 1 (Initiative)

1. Divide the class into suitable groups according to strength of students.
2. Make leader of each group.
3. Ask the student to tell the area of right angle Triangle.



4. Write this formula on black board i.e.

   Area = ½ (base x altitude)
5. Distribute different Right Angled Triangles of different size made of hard paper to each group.
6. Ask the students to find area of each triangle.
7. Write the measures of areas calculated by each group on black board.
8. Now make a figure of a triangle on black board which is not a right-angled and bend the attention of student to area of such type of Triangle.
9. It is exactly the time to tell the students that there are three case to find The area of any type of Triangle

Case No. 1 (Write on blackboard)

When measures of two sides and measure of one angle between these two sides in given, then

Area = $\Delta = \frac{1}{2}\, bc \sin\alpha$

Or $\Delta = \frac{1}{2}\, ac \sin\beta$

Or $\Delta = \frac{1}{2}\, ab \sin\gamma$

Where a, b, c represent sides and α, β, γ represent the angles.

Case No: 2

When measure of one side and measures of two angles are given, then

$$\Delta = \frac{1}{2} a^2 \frac{\sin\beta \sin\gamma}{\sin\alpha}$$

$$\Delta = \frac{1}{2} b^2 \frac{\sin\alpha \sin\gamma}{\sin\beta}$$

or $$\Delta = \frac{1}{2} c^2 \frac{\sin\alpha \sin\beta}{\sin\gamma}$$

Case No: 3

1. Area of triangle when measure of all the sides are given, then

$$\text{Area} = \Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

Where $S = \frac{1}{2}(a+b+c)$

This formula is also known as Hero's formula.

Activity No: 2

1. Divide the class into suitable groups.
2. Distribute acute angled Triangle of different sizes (with respect to sides) to each group.
3. Ask the student to measure angles and sides.
4. Guide the students in this activity.
5. Ask the student to calculate the area of triangle provided to each group.
6. Check the work of each group one by one and make necessary correction if needed.

Activity No: 3

1. Ask the students to find the area of a triangle ABC with the given data. (Write on black board)

   $b = 414 \quad c = 485 \quad \alpha = 49°47'$
2. Guide the students that here two sides and one angle is given, So we will use Case No.1 i.e. we will use

$$\Delta = \frac{1}{2} bc \sin\alpha$$

$$\Rightarrow \quad \Delta = \frac{1}{2}(414)(485) \sin(49°47')$$

$$\Rightarrow \quad \Delta = 76661.62 \text{ square units}$$

3. Here it is very important to tell the students that first analyse the data given in the question and then use the relevant formula.

Activity No: 4

1. Ask the students to find the area of triangles ABC in which
   $b = 47$, $\alpha = 60°\,25'$, $\gamma = 41°35'$
   (Write this on black board)
2. Write on black board that if two angles are given then third angle can be calculated by

$$\alpha + \beta + \gamma = 180°$$

$$\Rightarrow \quad \beta = 180° - (\alpha + \gamma)$$
$$\Rightarrow \quad \beta = 179°\,60' - (60°25' + 41°35')$$
$$\Rightarrow \quad \beta = 179°\,60' - (101°60')$$
$$\Rightarrow \quad \beta = 179°\,60' - 101°60'$$
$$\Rightarrow \quad \beta = 180° - 102°$$
$$\Rightarrow \quad \beta = 78°$$

3. Now tell the students that when measure of one side and two angles are given then it will be Case No:2

4. Guide the student to use the formula

$$\Delta = \frac{1}{2} b^2 \frac{\sin\alpha \sin\gamma}{\sin\beta}$$ (Demonstrate on black board)

   By putting values in the formula

$$\Delta = \frac{1}{2} (47)^2 \frac{\sin(60°25')\sin(41°35')}{\sin 78}$$

$$\Delta = 651.74 \text{ square unit}$$

Activity No:5

1. Ask the students to find the area of triangles ABC with given Data,

   $a = 276$, $b = 315$, $c = 524$ (Write on black board)

2. Ask the students to write this question on note book and guide then that here Hero's formula is used i.e.

$$\Delta = \sqrt{S\,(s-a)\,(s-b)\,(s-c)}$$

3. Tell the students that before using Hero's formula first find

$$S = \frac{1}{2}(a+b+c)$$ (Demonstrate on black board)

i.e. $S = \frac{1}{2}(276 + 315 + 524)$

$$\Rightarrow \quad S = \frac{1}{2}(1115)$$
$$\Rightarrow \quad S = 557.5$$

4. Using

$$\Delta = \sqrt{S\ (s-a)\ (s-b)\ (s-c)}$$

$\Rightarrow$ $\Delta = (557.5)\ 55.7-\ 276)(557.5\ -\ 315)(557.5-\ 524)$

$\Rightarrow$ $\Delta = 35705.89$ square units

Summary

(Summaries in the following way)

Actually area has a great importance in field of geometry. In field of trigonometry the area of triangle in the elementary classes give only the concept of area of right angled
Triangle i.e.

Area = ½ (base x perp)

While at higher secondary level we deals with any type of triangle whether right angled, acute angled or obtuse angled.

In short there are three cases through which we can find the area of triangle.

Self-Assessment:(Home Assignment)

Q No 1. Find the area of a triangle in which a= 450, b=220, c= 475

Q No 2. In a triangle ABC,

$\alpha = 47°$ $\beta = 65°$, $c = 21°$

Find the area of this triangle

Q No: 3. Find the area of a triangle with given data

a = 52.342, b = 29.572, c = 43.751

Q No: 4. **Fill in the blanks**

(i) **In Hero's formula**
Area =

(ii) **If in any triangle ABC,**
$a = 5$, $b = 2$, $\gamma = 30°$

Then area = ...

(iii) S = ...

Topic:
Definite Integral

Objective:

After studying this unit the students will be able to.

1. Define and evaluate definite integral.

2. Solve problems involving definite integrals.

Material Required:

Suitable numbers of copies of work sheets, Chalks, Blackboard, Duster etc.

Concept and Content:

The definite integral is one of the basic concept of mathematical analysis and it is a power research tool in mathematics, Physics, Mechanics, and other disciplines, Calculation of area bounded by curves, arc length, volumes, work, velocity, path length, moments of inertia and so forth reduce to the evaluation of a definite integral.

We can define definite integral as 'If f(x) is a continuos function on the integral [a,b] and if f(X) is any indefinite integral of f(x,.

Then $\int_a^b f(x)dx = f(b) - f(a)$ is called the **definite integral of f(x) between the limits a and b i.e.**

$$\int_a^b f(x)dx = \Big|\, F(x)\Big|_a^b = f(b) - f(a)$$

The symbol $\Big|\, F(x)\Big|_a^b$ is used to stand for the difference between the values of F(x) at $x = b$ and $x = a$ i.e. $F(b) - F(a)$

Properties of definite integral:

If f(x) and g(x) are two functions with 'a' and 'b' as a lower and upper limits, then,

i. $\int_a^a f(x)\,dx = 0$

ii. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ (change of limits)

iii. $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)dx$ **(c, being constant)**

iv. $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

v. $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx,$ **where $a \le c \le b$**

Example:

Find $\int_0^2 \frac{x^2}{x+1}dx$

Solution: $\int_0^2 \frac{x^2}{x+1}dx = \int_0^2 [x^2 - x + 1 - \frac{1}{x+1}]dx$

$$= \left| \frac{x3}{3} - \frac{x2}{2} + x - \ln(x+1) \right|_0^2$$

$$= [\frac{8}{3} - \frac{4}{2} + 2 - \ln(2+1)] - 0$$

$$= [\frac{8}{3} - \frac{4}{2} + 2 - \ln(2+1)] - 0$$

$$= \frac{8}{3} - 2 + 2 - \ln 3$$

$$\frac{8}{3} - \ln 3$$ Ans

Example:

Find

Solution:

Putting x-1 = t $\Rightarrow$ dx =dt

Now when x= -3 then, t=-2

When, x = -1 then, t = -4

Therefore, $$\int_{-3}^{-1}\frac{dx}{(x-1)^2} = \int_{-4}^{-2}\frac{dt}{t^2}$$

$$= \int_{-4}^{-2} t^{-2}\,dt$$

$$= - \frac{1}{t}$$

$$= -(-\frac{1}{2}+\frac{1}{4})$$

$$= \frac{1}{4} \text{ Ans}$$

Example

Evaluate $\displaystyle\int_{0}^{\pi/3} \frac{dx}{1 - \sin x}$

Solution $\displaystyle\int_{0}^{\pi/3} \frac{dx}{1 - \sin x}$

$$= \int_{0}^{\pi/3} \frac{(1 + \sin x)\, dx}{(1 - \sin x)(1 + \sin x)}$$

$$= \int_{0}^{\pi/3} \frac{(1 + \sin x)\, dx}{(1 - \sin^2 x)}$$

$$= \int_{0}^{\pi/3} \frac{(1 + \sin x)\, dx}{\cos^2 x}$$

$$= \int_{0}^{\pi/3} (\sec^2 x + \sec x \tan x)\, dx$$

$$= \Big|\tan x + \sec x\Big|_{-4}^{3}$$

$$= \tan \pi/3 - \tan 0 + \sec \pi/3 - \sec 0$$

$$= \sqrt{3} - 0 + 2 - 1 = \sqrt{3} + 1 \quad \text{Ans}$$

Example:

Evaluate, $\int_{-3}^{3} \frac{dx}{x^2+9}$

Solution: $\int_{-3}^{3} \frac{dx}{x^2+9} = \left| \frac{1}{3} \tan^{-1} \frac{x}{3} \right|_{-3}^{3}$

$$= \frac{1}{3} \left| \tan^{-1} \frac{x}{3} \right.$$

$$= \frac{1}{3} \left[\tan^{-1} \frac{3}{3} - \tan^{-1} \frac{-3}{3} \right]$$

$$= \frac{1}{3} \left[\tan^{-1}(1) - \tan^{-1}(-1) \right]$$

$$= \frac{1}{3} \left[\frac{\pi}{4} - \left(\frac{-\pi}{4}\right) \right]$$

$$= \frac{1}{3} \left[\frac{\pi}{4} + \frac{\pi}{4} \right]$$

$$= \frac{1}{3} \frac{\pi}{2}$$

$$= \frac{\pi}{6} \textit{ Answer}$$

(Methodology)

Activity No 1. (Initiative):

1. **Divide the class in to pair groups.**
2. **Distribute copies of work sheets to each group.**
3. **Give instruction to students to read the questions carefully and answer them as directed.**
4. **Visit every group monitor and guide him or her.**
5. **Tell them to discuss each other and then give answer of question asked.**
6. **Now collect the work sheets from each group.**
7. **Now step froward to your topic to be delivered keeping in mind the approach of students.**
8. **Tell the simple definition of "Definite integral" i.e.**
   "If f(x) is a continuous function on interval [a,b] and if F(x) is any indefinite integral of f(x) then,

$$\int_a^b f(x)dx = \Big| F(x) \Big|_a^b = f(b) - f(a)$$

Tell the students that if f(x) and g(x) are any two functions with a and b as lower and upper limits, then following are the properties of definite integral,

i. $\int_a^a f(x)\,dx = 0$

ii. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ (change of limits)

iii. $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)dx$ **(c, being constant)**

iv. $\int_a^b [f(x) \pm g(x]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

v. $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx,$ **where $a \leq c \leq b$**

NOTE

A sample of work sheet is given on the last page of the module.

Activity No2:

1. **Write the following question on black board.**

   " Find", $\int_0^2 \frac{x^3}{x+1}\,dx$

2. **Tell them that as the degree in Numerator is greater than the degree of polynomial given in the denominator, so we will take help by long division. i.e.**

$$
\begin{array}{r|l}
x+1 & \cancel{x^3} \quad (x^2-x+1 \\
 & \underline{-\cancel{x^3} \pm x^2} \\
 & -\cancel{x^2} \\
 & \underline{\mp \cancel{x^2} \mp x} \\
 & \cancel{x} \\
 & \underline{-\cancel{x} \pm 1} \\
 & -1
\end{array}
$$

3. **It can be written as** $\frac{x^3}{x+1} = x^2-x+1 - \frac{1}{x+1}$

4. **Tell them that now we are in the position to find definite integral.**

5. Therefore, $\int_0^2 \frac{x3}{x+1}dx = \int_0^2 [x2 \quad x+1 - \frac{1}{x+1}]dx$

$$= \left| \frac{x3}{3} \quad \frac{x2}{2} + x - \ln(x+1) \right|_0^2$$

$$= \left[\frac{8}{3} \quad \frac{4}{2} + 2 - \ln(2+1)\right] - \left[\frac{0}{3} - \frac{0}{2} + 0 - \ln(0+1)\right]$$

Tell the students that they note this problems on their notebook and remove their difficulties.

Activity No3:

1. **Write the question,**

Find, $\int_{-3}^{-1} \frac{dx}{(x-1)^2}$ **(Substitution method)**

Solution

2. **Tell the students that here we will use substitution method. i-e**

Let

x-1 =t $\Rightarrow$ dx = dt

now when x= -3, then, t = -2

and when x= -1, then, t = -4

3. **So above given integral will be totally changed from 'x' variable into 't' variable. This point is very logical, clear this point is very carefully and discuss it thorowly.**

4. **Now,**

$$\int_{-3}^{-1} \frac{dx}{(x-1)^2} = \int_{-4}^{-2} \frac{dt}{t^2}$$

$$= \int_{-4}^{-2} t^{-2}\, dt$$

$$= - \left| \frac{1}{t} \right|_{-4}^{-2}$$

$$= \quad -(-\frac{1}{2}+\frac{1}{4})$$

$$= \quad \frac{1}{4} \text{ Ans}$$

5. **Now ask the students to tell the difficulties which they felt during the solution of problem.**
6. **Provide open choice to students to ask questions, so that the topic may be perfectly cleared and make it interesting through mutual discussion.**
7. **Tell the students to note it on their notebooks.**

Activity No 4:

1. **Tell the students to write the following question on their note book and write your self on black board.**

Evaluate, $\int_{0}^{\frac{\pi}{3}} \frac{dx}{1-\sin x}$

SOLUTION:

2. **Just for one or two minutes, allow the students to think about, what should be the first step.**

3. **Keeping the views of some students in mind, tell them that here as there are two terms in the denominator, so we can use the method of rationalization.**

4. **So,**

$$\int_0^{\pi/3} \frac{dx}{1-\sin x} = \int_0^{\pi/3} \frac{(1+\sin x)\,dx}{(1-\sin x)(1+\sin x)}$$

$$\int_0^{\pi/3} \frac{(1+\sin x)\,dx}{(1-\sin^2 x)}$$

$$\int_0^{\pi/3} \frac{(1+\sin x)\,dx}{\cos^2 x}$$

$$\int_0^{\pi/3} (\sec^2 x + \sec x \tan x)\,dx$$

$$= \Big|\tan x + \sec x\Big|_0^{\pi/3}$$

$$\tan \pi/3 - \tan 0 + \sec \pi/3 - \sec 0$$

$$\sqrt{3} - 0 + 2 - 1 = \sqrt{3} + 1 \text{ Ans.}$$

5. **Now, remove the difficulties if needed.**

Self-Assessment: -

Assign the following task for Homework.

Q No.1: **Find,** $\int_{1}^{3} \frac{x}{1+x^2}\,dx$

Q No. 2: **Define definite integral**

Q No.3: **Evaluate,**

$$\int_{0}^{1} (5x^4 + e^x)\,dx$$

Q No.4: **Find,**

$$\int_{-\pi/2}^{\pi/2} \cos x\,dx$$

Q No.5. **Do as directed,**

(i) **The value of ,** $\int_{2}^{3} \frac{1}{x}\,dx$ **is**

(a) $\log \frac{2}{3}$ (b) $\log \frac{3}{2}$ (c) $\log \frac{1}{6}$ (d) non of a, b, c

(ii) $\int_{0}^{2} e^x\,dx$ gives the value

(a) **0** **(b)** **1** **(c)** **2** **(d)** **non of a, b, c**

(iii) **The value of** $\int_{0}^{1} x\,dx =$ ....

(iv) $\int_{\pi}^{0} \sin x\,dx =$ ....

(v) $\int_{1}^{2} 2x\,dx$ is equal to to ....

Ans of Q.5 (i) b (ii) a (iii) $\frac{1}{2}$ **(iv) 2 (v) 3**

Sample of work sheet

1. **In the method of integration by parts, we use the formula:**

   $\int uv dx =$...............

2. $\int \frac{1}{x} dx =$................

3. $\int$ cosecx cot x dx = ...

4. **Write the power rule for anti –derivative .**

5. **(Matching type items)**

Column	Column 2
(i) $\int x^n dx$ is :	(a) $\frac{3^x}{\ln 3} + c$
(ii) $\int e^{ax} dx$ is :	(b) $\frac{e^{ax}}{a} + c$
(iii) $\int x^{1/7} dx$ is :	(c) $\frac{x^{n+1}}{n+1} + c$
(iv) $\int 3^x dx$ is :	(d) $\frac{7}{8} x^{8/7} + c$
	(e) nx^{n-1}

Answers Q No ;5 (i) →c (ii)→(b) (iii)→ d (iv) →(a)

Q No ;2 lnx , Q No ;3 – cosecx

Q No ;4 $\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$